1 Background and Outline

A great deal of work in algebraic topology has revolved around settling the Kervaire invariant one problem: when are framed manifolds not frame bordant to spheres. Kervaire described a generalization of Pontryagin's work [**?**], using it to show the existence of a non-smoothable 10-manifold. Kervaire-Milnor built on this and Milnor's exotic 7-spheres to link the number of exotic smooth structures on the n-sphere to the n^{th} stable homotopy group of spheres [**?**], [**?**], showing that the subgroup of the stable stems corresponding to framed homotopy spheres has index at most two in the stable stems and that the quotient is generated by a manifold of Kervaire invariant one. This was distilled by Browder to a question of the survival of a particular family of elements in the Adams spectral sequence [**?**].

Hopkins, Ravenel, and I solved this problem in 2009, using techniques and computations in C_8-equivariant homotopy theory [**?**]. There is a rich literature on equivariant homotopy theory for finite and compact Lie groups G. While basic structural results about spectra and ring spectra are known, much is unexplored. In general, computations are difficult and few are known. Our solution to the Kervaire invariant one problem used two primary tools: an analysis of equivariant commutative ring spectra and the norm functor, and the computational tool of the slice filtration. Warm-up computations with the slice spectral sequence have a tantalizing connection to the theory of topological modular forms: there are equivariant spectra refining certain Galois extensions of topological modular forms with level structure. The computations also have the structure that an equivariant commutative ring spectrum would have. This frames two motivating conjectures which broadly define the scope of all of the projects below.

Motivating Conjecture 1. *Moduli problems arising from algebraic geometry taking values in commutative ring spectra with a G-action have natural refinement to a moduli problem taking values in genuine G-equivariant commutative ring spectra.*

By analogy with "derived algebraic geometry" being algebraic geometry with structure sheaves enriched in commutative ring spectra, we call such moduli problems producing genuine equivariant commutative ring spectra "equivariant derived algebraic geometry".

Motivating Conjecture 2. *The equivariant spectra arising in equivariant derived algebraic geometry have slices determined by underlying algebro-geometric data.*

Equivariant Tools from the Kervaire Invariant One Problem

Building on our previous collaborative efforts to understand the homotopy groups of the Hopkins-Miller spectra, Hopkins, Ravenel, and I proved the following theorem [**?**].

Theorem 1. *There are manifolds of Kervaire invariant one only in dimensions* 2*,* 6*,* 14*,* 30*,* 62*, and possibly* 126*.*

To prove this, we lifted Browder's result to the Adams-Novikov spectral sequence and then recast it as a question in C_8-equivariant homotopy theory which we solved using the norm and the slice filtration described below.

The rigidification of the problem to one in genuine equivariant homotopy added extra structure but also more complexity. One of the primary difficulties is that homotopy classes of maps between two genuine G-spectra are naturally G-Mackey functors: if T is a finite G-set, then we can define a richer structure out of the G-equivariant homotopy classes of maps

$$\underline{[X,Y]}(T) := [T_+ \wedge X, Y]^G \cong [X, T_+ \wedge Y]^G,$$

using the self-duality of finite G-sets [**?**]. The first formulation is obviously contravariant in T; the latter is covariant. This describes a Mackey functor. Since this structure is visibly natural in X and in Y, we see that all of our constructions in the genuine equivariant context are naturally Mackey functor valued. In particular, all of the work described below will interface in some way with Mackey functors and the transfer, and keeping this in mind is a helpful guide.

The norm

The homotopy groups of equivariant commutative rings have more structure than just Mackey functors. The multiplication on them endows the homotopy Mackey functors with a commutative multiplication, making them commutative Green functors. However, we also have multiplicative versions of the transfer called "norm maps" [**?**], making the homotopy groups into a Tambara functor. This structure is reflected in spectra.

Theorem 2. *There is a homotopically meaningful symmetric monoidal functor*

$$N_H^G \colon \mathcal{S}p^H \to \mathcal{S}p^G$$

which is the left adjoint to the forgetful functor on equivariant commutative ring spectra.

The construction is straightforward: if we have an H-spectrum, then we smash together $|G/H|$-copies and let G act by permuting the coordinates and extending the H-action. In other words, we mirror the ordinary induction functor but multiplicatively. Since the smash product is the coproduct for commutative ring spectra, the classical arguments for induction imply that the norm is the left adjoint to the forgetful functor. All of the work (in fact, much of Appendix B in [**?**]) goes into showing that this is homotopically meaningful.

The norm has proved extremely helpful in structuring an analysis of multiplicative structures in equivariant homotopy, starting with my localization result with Hopkins [**?**].

Theorem 3. *If the category of E-acyclics is closed under norms in the sense that if Z is an E-acyclic then $N_H^G i_H^* Z$ is also an E-acyclic, then the Bousfield localization $L_E(-)$ preserves equivariant commutative ring spectra.*

In general, equivariant Bousfield localization need not preserve commutative rings, and it is exactly which norms fail to preserve the category of acyclics that provide the obstruction.

Building on this observation, Blumberg and I studied the kinds of coherently commutative multiplications that can occur in equivariant spectra [**?**]. We defined a class of "N_∞ operads", the spaces of which have a free Σ_n action and for which are all underlain by contractible spaces. This language provides a nice generalization of the localization result above: all equivariant Bousfield localizations of commutative ring spectra have a coherently commutative multiplication. It also provides ways to show how transfers and norms arise operadically. Extension of this work is described below in Section 2.

Finally, the norm functor provides the backbone of our new model for topological Hochschild homology [?]. We *define* the topological Hochschild homology functor to be S^1-equivariant norm, the left adjoint to the forgetful functor from S^1-equivariant commutative ring spectra to ordinary commutative ring spectra. This allows several quite computable models for relative versions of topological Hochschild homology: one starts with H-equivariant commutative rings for H a finite subgroup of S^1 and produces an S^1-equivariant spectrum and the other produces an S^1-equivariant model for THH in the category of R-modules for some fixed commutative ring spectrum R. Explorations of this is described in Section 4.

The slice filtration

The slice filtration is an equivariant filtration generalizing work of Dugger for $G = C_2$ and loosely modeled on the motivic analogue due to Voevodsky [?], [?, ?, ?]. The C_2-equivariant version was studied by Hu-Kriz in their analysis of the Landweber-Araki Real bordism spectrum $MU_{\mathbb{R}}$ [?], and they provide a nice connection between it and the motivic version [?].

For spectra like $MU^{(G)}$, the slice filtration is completely determined by the underlying homotopy. The spectrum underlying $MU^{(G)}$ is $MU^{\wedge|G|/2}$, and so the homotopy ring is the $|G|/2$-fold tensor power of the Lazard ring. All of the generators refine to equivariant maps from representation spheres to $MU^{(G)}$, and the slice associated graded is simply a wedge of integral Eilenberg-Mac Lane spectra index by these $RO(G)$-graded monomials. This gives us a very useful tool for computing the Mackey functor of maps from X into $MU^{(G)}$, since the E_2 term is simply a sum of $RO(G)$-graded Bredon homology groups of X. For X a sphere, this simply a sum of $RO(G)$-graded Bredon homology groups of a point.

If M is the $MU^{(G)}$ module formed from $MU^{(G)}$ by killing some collection of the refinements of the underlying polynomial generators, then we have an analogous slice spectral sequence computing equivariant M-cohomology. There is an analogue of Quillen's idempotent G-equivariantly, producing a spectrum $BP^{(G)}$, a summand of $MU^{(G)}$, and we can form spectra $BP_G\langle m\rangle$ out of $BP^{(G)}$ by killing all of the generators analogous to v_{m+1} and above. Computations with these are essentially completely determined by those of $MU^{(G)}$, but for all of these quotients, the slice spectral sequence terminates after a finite stage. From the point of view of chromatic homotopy, cursory computations show that the spectra $BP_G\langle m\rangle$ are underlain by ordinary spectra of height $2^{n-1}m$. We explore computations with these in Section 3.

Equivariant Derived Algebraic Geometry

Before tackling attempting the Kervaire problem, we did several warm-up computations with the slice spectral sequence, augmenting the foundational work of Dugger on $K_{\mathbb{R}}$. The slice spectral sequence can be used to compute the homotopy groups of the Real Johnson-Wilson spectra $E\mathbb{R}(n)$, defined by Hu-Kriz [?] and studied by Kitchloo-Wilson [?, ?]. The slice spectral sequence here is extremely simple and entirely determined by that of $MU_{\mathbb{R}}$. As a final, more detailed computation, we analyzed $BP_{C_4}\langle 1\rangle$ and the analogous $E_{C_4}(1)$ [?]. These are underlain by height 2 theories, and for $E_{C_4}(1)$, the fixed and homotopy fixed points agree.

The slice spectral sequence for $E\mathbb{R}(2)$ and for the connective analogue $BP_{\mathbb{R}}\langle 2\rangle$ coincide with work of Mahowald-Rezk on the homotopy groups of the spectrum of topological modular forms with a $\Gamma_0(3)$ structure $TMF_0(3)$ and its connective cover $tmf_0(3)$ [?]. Additionally, the homotopy ring, as an $RO(C_2)$-graded object has a natural Tambara functor structure inherited from that of $MU_{\mathbb{R}}$. This suggests that there could be a genuine C_2-equivariant commutative ring spectrum which is underlain by the spectrum of topological modular forms with a $\Gamma_1(3)$-structure and giving these computations.

This guess is strengthened by the computation for $BP_{C_4}\langle 1\rangle$. The slice computation shows again that we have norms, and the computation is the same as the computation of Mahowald and of Behrens-Ormsby of the homotopy groups of the spectrum of topological modular forms with a $\Gamma_0(5)$ structure [**?**]. So again, there seems to be a genuine equivariant commutative ring spectrum rigidifying the naïve equivariant spectrum produced by the moduli stack of elliptic curves.

In both cases, there is an underlying algebraic geometry moduli problem which has a solution in so-called "derived stacks", stacks with a refinement of their structure sheaf to a sheaf of commutative ring spectra. In the periodic cases, this is the Goerss-Hopkins-Miller sheaf $\mathcal{O}^{top}$ on the moduli stack of elliptic curves [**?**], [**?**], and in the connective cases, this is Lawson and my extension of $\mathcal{O}^{top}$ over the log-étale site [**?**].

Derived algebraic geometry is a central focus in modern algebraic topology. Instead of working with ordinary commutative rings, we do algebraic geometry using commutative ring spectra. Thus ordinary algebraic geometry notions like elliptic curves, abelian varieties, and moduli spaces have "derived" versions in which the corresponding structure sheaves are sheaves of commutative ring spectra or modules over them. Based on the expansive and pioneering work of Lurie [**?**], authors like Behrens-Lawson, Lawson-Hill, Lawson-Naumann, Meier-Mathew, Stojanoska, and others have built new and interesting commutative ring spectra and maps between them by producing derived sheaves [**?**], [**?**], [**?**], [**?**], [**?**].

Since the automorphisms of a point act on the value of the sheaf on that point, we get naturally commutative ring spectra with actions of finite groups. By definition, the values of such a sheaf always land in naïve equivariant commutative rings. The full strength of Conjecture 1 is that we can evaluate these sheave on other equivariant commutative ring spectra, producing theories that would otherwise be inaccessible.

In the examples described above, the underlying data is simply the classical moduli problems together with the action of the finite group. Moreover, the theories produced classically by evaluating on Spec of rings are Landweber exact theories, and therefore of the form a localization of $BP_G\langle n\rangle$ described above. This is the computational heart of Conjecture 2, since it would show that these objects are built out of computationally simple pieces, just as the sheaves in derived algebraic geometry are built out of quotients of MU.

Outline

Conjectures 1 and 2 are very difficult conjectures. Both of these require a great deal of the non-equivariant derived algebraic geometry machinery to be ported over to an equivariant setting. The projects described below fit into three distinct classes, each of which go into proving these conjectures and each of which also is of independent interest.

Section 2 expands my work with Blumberg to look at equivariant commutativity and implementing the Goerss-Hopkins obstruction theory in eDAG. Since this forms the basic tool in producing sheaves of commutative ring spectra, understanding the equivariant version is key. On the other hand, norms and transfers are fundamental objects of study equivariantly, and these projects explain how they fit in.

Section 3 explores computational aspects. A detailed analysis of equivariant Landweber exact theories and $MU^{(G)}$-modules is both tractable and directly relevant to Conjecture 2. Moreover, these computations describe an interesting filtration on the homotopy groups of spheres. On the purely algebraic side, any variant of the Goerss-Hopkins obstruction theory will need a solid foundation of algebraic geometry over a Green or Tambara functor. Especially in the $RO(G)$-graded context, essentially nothing is known here.

Finally, in Section 4, I outline the consequences of this and related work in topological Hochschild homology. All of the underlying algebra issues which arise in proving Conjecture 1 arise in various guises here, making this a useful proving ground for conjectures about equivariant commutative rings. Algebraic K-theory itself is a fundamental object of study in topology, number theory, and algebraic geometry, so anything we can say computationally is of intrinsic interest.

2 Flavors of Commutativity

2.1 N_∞ Operads

In the equivariant setting, there are many distinct homotopical operadic notions of "commutative ring spectrum". Blumberg and I studied these intermediate notions of commutativity [**?**]. We defined an "N_∞ operad" to be a G-operad for which the n^{th} space is a universal space for a family of subgroups of $G \times \Sigma_n$ and for which Σ_n itself acts freely.

The classification of homotopy types of N_∞ operads is equivalent to which norms algebras over the operad necessarily have. Both are determined by particular subcategories of the categories of finite H sets as H ranges over all subgroups of G. If H is a finite group and T is a finite H-set of cardinality n, then an identification of T with the set of n elements produces a homomorphism $H \to \Sigma_n$. If we let Γ_T be the graph of this homomorphism, viewed as a subgroup of $H \times \Sigma_n \subset G \times \Sigma_n$, then we say that T is "admissible" for an N_∞ operad $\mathcal{O}$ if $\mathcal{O}_n^{\Gamma_T}$ is contractible. The full subcategories of admissible H-sets as H ranges over all of the subgroups of G defines a contravariant functor from the orbit category of G to the category of small categories.

The admissible sets are closed under subobjects, under products, under disjoint union, and under induction by other admissible sets. We call a sub-category-valued coefficient system of the coefficient system of finite sets an "indexing system", and these obviously form a poset under inclusion. Blumberg and I proved that this is a complete invariant of N_∞ operads, up to homotopy.

Theorem 4. *The functor $\underline{\mathcal{C}}(-)$ taking an N_∞ operad to its indexing system of admissible sets is a fully-faithful inclusion of the homotopy category of N_∞ operads into the poset of indexing systems.*

Unfortunately, we were unable to show that this functor is essentially surjective. Work of Guillou-May provides a way to construct a symmetric sequence of the right homotopy type; it is not clear that the result is an operad [**?**]. This is a combinatorially accessible question, requiring only an understanding of symmetric groups and the multiplication therein, making this problem approachable by students.

Straightforward Conjecture 3. *The functor $\underline{\mathcal{C}}$ is essentially surjective.*

One of the conceptual difficulties in proving that $\underline{\mathcal{C}}$ is essentially surjective is that we have no way to realize the least upper bound of two elements $\underline{\mathcal{C}}(\mathcal{O}_1)$ and $\underline{\mathcal{C}}(\mathcal{O}_2)$. Similarly, one of the issues in identifying exactly how the $\mathcal{O}$-algebra structure affects computations is in understanding when an N_∞ operad acts on itself by N_∞ maps. Operadic literature describes this as the operad "interchanging" with itself, and it is controlled by the Boardman-Vogt tensor product [**?**]. The inclusions into the tensor product of the factors also shows that if the tensor product is again an N_∞ operad, then it is bigger than both factors.

Conjecture 4. *1. Every homotopy type of N_∞ operad is represented by an operad that interchanges with itself.*

2. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are N_∞-operads, then $\mathcal{O}_1 \otimes \mathcal{O}_2$ is again an N_∞ operad and $\underline{\mathcal{C}}$ of it realizes the join in the poset.

The admissible sets also describe which norms an algebra over an N_∞ operad possesses. In both spaces and spectra, we can let $N^T(-)$ denote the functor "apply the symmetric monoidal power indexed by T", as in [**?**]. In G-spectra, this is the smash product over T of $i_H^* R$, the norm associated to T described in the introduction. In G-spaces, this is the space of H-equivariant maps $Map^H(T, i_H^* R)$.

Theorem 5. *If $\mathcal{O}$ is an N_∞ operad and R is an $\mathcal{O}$-algebra, then for each admissible H-set T, the operad $\mathcal{O}$ provides a contractible space of maps*

$$N^T(i_H^* R) \to i_H^*(R).$$

These maps are basic objects of study in both spaces and in spectra. In spaces, these give transfer maps, producing the infinite loops of the transfer maps when $\mathcal{O}$ is the little discs operad on a universe U. In spectra, these maps are essentially unstudied beyond the original work of Greenlees-May on norms in commutative ring spectra and in Hill-Hopkins-Ravenel [**?**], [**?**].

The functoriality of Ω^∞ and classical work of May shows that the zero space of an $\mathcal{O}$-algebra in spectra is a $\mathcal{O}$-ringed spaced [**?**]. It makes sense then to ask about GL_1 of an $\mathcal{O}$-algebra. Using the poset-structure, if U is any universe for which the little discs operad is less than or equal to $\mathcal{O}$, we can deloop producing a G-spectrum $\mathfrak{gl}_1$ indexed by U. We expect a kind of logarithmic behavior.

Straightforward Conjecture 5. *If R is an $\mathcal{O}$-algebra, then the transfers in $GL_1(R)$ and in $\mathfrak{gl}_1(R)$ arise from the associated norm maps in R.*

2.2 N_∞-Categories

The theory of N_∞ operads shows that norms are extra structure added to a multiplication that is commutative up to all higher homotopy. This begs the question: what structure does the category of modules over an algebra over an N_∞ operad have? Classically, if R is an E_∞-ring spectrum, then the category of R-modules is symmetric monoidal. The standard argument relies on replacing R with an equivalent commutative ring spectrum, where the result is immediate.

Straightforward Conjecture 6. *If $\mathcal{O}$ is an N_∞-operad and if R is an $\mathcal{O}$-algebra, then*

1. *the category of R-modules is a symmetric monoidal category*

2. *$R - \mathcal{M}od$ has internal norms parameterized by the admissible sets for $\mathcal{O}$.*

The first point should be provable by a slight modification of the standard techniques. The linear isometries operad $\mathcal{L}$ for a trivial universe is always a naïve N_∞ operad. By restriction of structure, any $\mathcal{O}$-algebra is a $\mathcal{L}$-algebra, and if we build a new model of $\mathcal{L}(1)$-spectra, then our original algebra R becomes a commutative ring spectrum in this structure. We can then apply an EKMM style analysis [**?**].

The second point is completely unobserved and unexplored in the literature. If $\mathcal{O}$ interchanges with itself, then by Theorem 5, for any admissible G-set T, we have an $\mathcal{O}$-algebra map $N^T R \to R$. By functoriality of N^T, if M is an R-module, then $N^T M$ is an $N^T R$-module. If we base change along the map $N^T R \to R$, then we can define an internal norm:

$$N_R^T(M) = N^T(M) \wedge_{N^T(R)} R.$$

This is extra structure on the category of R-modules, and it is natural is R. This also has immediate and useful consequences for things like localization, as out of this observation, we can prove the following.

Conjecture 7. *If $\mathcal{O}$ is an N_∞ operad and if R is an $\mathcal{O}$-algebra in spectra, then the R-Bousfield localization $L_R(-)$ preserves $\mathcal{O}$-algebras.*

We use something like these relative norms in our model of relative THH described below, but otherwise, this seems to be new, even in algebra. This has several interesting consequence, especially when the operad is neither naïve nor genuine. Most notably, if A is a $\mathcal{O}$-algebra in R-algebras, then we can ask how the norms of A interact with the norms of R.

There should also be a higher categorical approach to the sort of structure we are observing. In this case, a symmetric monoidal category is a kind of quasi-category fibered over the nerve of finite sets [**?**]. For N_∞ operads in which the admissible H-sets are determined by those of G, this gives an natural quasi-categorical extension.

Conjecture 8. *If R is an $\mathcal{O}$-algebra, then the category of R-modules is a quasi-category fibered over the nerve of the category of admissible G-sets.*

For operads where the category of admissible sets for a subgroup H is not determined by those for G, it is not yet clear how even to formulate this conjecture.

2.3 Obstruction Theory and the Dyer-Lashof Algebra

The Goerss-Hopkins obstruction theory for algebras over an operad has been the primary tool used to produce commutative ring spectra out of algebraic data [**?**]. A huge amount of this breaks down in the equivariant context.

Hard Goal 9. *Build the G-equivariant Goerss-Hopkins obstruction theory and apply it to classical results like the Hopkins-Miller theorem.*

We can of course formally mirror the arguments of Goerss-Hopkins and produce an analogous obstruction theory; identifying what it produces and computing with it are another matter. There are two related computational difficulties: determining and computing the relevant André-Quillen cohomology groups and computing the particular Dyer-Lashof algebra. The determination of the André-Quillen cohomology groups is a purely algebraic question: how do we compute this Quillen cohomology in commutative Green functors and how do we incorporate the norms? I will return to this below in Section 3.1. The questions about the Dyer-Lashof algebra are more topological.

There is are Dyer-Lashof algebras for each homotopy type of N_∞ operad. Classically, the Dyer-Lashof algebra in E-theory arises from the E-homology of classifying spaces $B\Sigma_n$ as n-varies (and the operadic multiplication). Similarly, with an N_∞ operad $\mathcal{O}$, the Dyer-Lashof algebra for $\mathcal{O}$ arises from the classifying spaces $\mathcal{O}_n/\Sigma_n$.

We can conceptually understand these spaces for each n before passage to Σ_n orbits, as this is exactly what is recorded in Theorem 5. For each finite H-set T of cardinality n, we have an H-equivariant n-ary operation

$$\times_T \colon N^T i_e^* R \to R.$$

These commute up to higher homotopy, meaning that we have cells identifying the effect of rotation with what we start with, and moreover, if T and T' are two finite H-sets which become isomorphic when restricted to K, then there are cells making the operations $i_K^* \times_T$ and $i_K^* \times_{T'}$ homotopic. For $G = C_2$ and $n = 2$, the explicit model of infinite spheres in various representations of $C_2 \times \Sigma_2$ allow a clear example. Passing to $n = 4$ however already becomes much more difficult, as we have to introduce additional cells that realize the isomorphism $[C_2] \times [C_2] \cong 2[C_2]$.

By asking a slightly more general question, we can tie this to the tom Dieck splitting and the fixed points for the free $\mathcal{O}$-algebra in spaces.

Straightforward Conjecture 10. *Let $\mathcal{O}$ be an N_∞ operad, and let $\mathbb{P}_{\mathcal{O}}$ be the free $\mathcal{O}$-algebra functor in spaces. We have a canonical weak equivalence*

$$\big(\mathbb{P}_{\mathcal{O}}(X)\big)^G \simeq \coprod_{T \in \underline{\mathcal{C}}(\mathcal{O})(G/G)} E\,\mathrm{Aut}_G(T) \times_{\mathrm{Aut}_G(T)} Map^G(T, X),$$

where $\mathrm{Aut}_G(T)$ is the group of G-equivariant automorphisms of T, and $Map^G(T, X)$ is the space of G-equivariant maps from T to X.

This is the analog of the infinite loop space version of the tom Dieck splitting. Here, however, we see much more transparently the roles played by the various transfers indexed by various subgroups. It also explicitly shows this as a “free algebra”, but we have all possible operations treated on equal footing. Letting X be a point, we recover the desired orbit space for $\mathcal{O}_n/\Sigma_n$ and the Dyer-Lashof algebra.

Classically, few Dyer-Lashof algebras are completely known. Dyer and Lashof focused on the case of ordinary homology [?], [?]; McClure focused on complex K-theory [?], and work of Rezk looks $K(n)$-locally at Lubin-Tate spectra [?]. All of these examples have been extremely useful, and the first two should be approachable equivariantly.

Goal 11. *Determine the Dyer-Lashof algebras for $K_{\mathbb{R}}(1) = K_{\mathbb{R}}/2$, for the $K(1)$ (viewed as a spectrum with a trivial action), and for ordinary cohomology with coefficients in a Mackey field.*

The first two examples should give information about ways we can mirror Hopkins’ approach to $K(1)$-local E_∞ [?]. The last two should be essentially classical, just as Oruç showed that the equivariant Steenrod algebra over a Mackey field is classical [?].

3 Equivariant computations

3.1 Green algebraic geometry

One of the principle difficulties in extending derived algebraic geometry to the equivariant setting is a lack of good algebraic models. The homotopy groups of an algebra over an N_∞ operad are commutative Green functors with norms (exactly which norms depends on the admissible sets for the operad). Rather than having a different model for each flavor of commutative Green functor (the approach of authors like Nakaoka [**?**]), we instead emphasize that the norms are additional structure on an underlying commutative Green functor and build an algebraic geometry which reflects this. Additionally, we want our theory to both inform and be informed by equivariant ring spectra. In particular, we should have not just a set of maps between two Green schemes but rather a set-valued coefficient system (this is the effect of $\underline{\pi_0}$ on the space of maps between G ring spectra). I have a candidate for Green algebraic geometry which does include all of the desired properties. For completeness, I will sketch this while describing needed additional work.

A commutative Green functor $\underline{R}$ is, in particular, a contravariant, product preserving functor from the category of finite G-sets to the category of commutative rings. Applying $\mathrm{Spec}(-)$ then produces a covariant functor from the category of finite G-sets to the category of schemes which sends disjoint unions to disjoint unions. Call this composite functor $\underline{\mathrm{Spec}}(-)$. This produces the underlying co-coefficient system of spaces locally describing our Green schemes.

At this point, we have only used the underlying coefficient system of commutative rings. The transfer maps in a commutative Green functor are required to satisfy an additional condition: if $S \to T$ is a map of finite G-sets, then the transfer map $\underline{R}(S) \to \underline{R}(T)$ is required to be a map of $\underline{R}(T)$-modules (where the source is an $\underline{R}(T)$-module via the restriction). This translates to the additional data of maps of sheaves of modules over $\underline{\mathrm{Spec}}(\underline{R})$. This formulation obviously extends to a notion of a $\underline{\text{scheme}}$[1], which is just a covariant functor from the orbit category to schemes together with $\underline{\mathcal{O}_X}$-module maps playing the role of the transfer.

Theorem 6. *The functor* $\underline{\mathrm{Spec}}(-)$ *is a fully-faithful embedding of the category of commutative Green functors into the category of* $\underline{\textit{schemes}}$*.*

As written, this just produces a set of maps between any two $\underline{\text{schemes}}$. However, if we remember the restriction maps, then we can say that

$$\underline{\mathrm{scheme}}(\underline{X}, \underline{Y})(G/H) = \underline{\mathrm{scheme}}(i_H^*\underline{X}, i_H^*\underline{Y}).$$

Put another way, the forgetful functor from G-Green functors to H-Green functors has a left adjoint L_H^G. For formal reasons, the following should be true.

Straightforward Conjecture 12. *The left adjoint* L_H^G *extends to a functor on* $\underline{\textit{schemes}}$*, and the coefficient system of maps can be recovered from this functor.*

In this formulation, computations are both possible and weird. We content ourself to flavors of $\mathbb{A}^1$ and $\mathbb{P}^1$. If T is a finite G-set, then the functor

$$\underline{R} \mapsto \underline{R}(T)$$

[1]following a notational practice established by Lewis for other algebraic structures built out of Mackey functors

on commutative Green functors is representable by a scheme $\mathbb{A}^1_T$. Just as in the classical case, this is actually a group scheme, representing the underlying abelian group. Moreover, if $S \to T$ is a map of finite G-sets, then there are maps $\mathbb{A}^1_S \leftrightarrows \mathbb{A}^1_T$ representing the restriction and transfer maps in the Green functor (so in particular, $T \mapsto \mathbb{A}^1_T$ is a Mackey scheme).

With so many flavors of $\mathbb{A}^1$, we have a similar number of kinds of $\mathbb{P}^1$. This seems to be slightly less well-behaved. In general, the symmetric powers of a projective Mackey functor need not be even flat (the symmetric square of $\underline{A}_{C_2}$ demonstrates this for $G = C_2$), so the resulting object will not necessarily be flat. Even so, it is not difficult to explicitly write down these objects.

These are of course just warm-up exercises. More interesting is determining how many of the classical constructions go through, in particular, when we can use algebraic data to completely describe deformations.

Hard Goal 13. *Understand the homotopically meaningful relative cotangent complex for a map of* <u>*schemes*</u> *and the extent to which it controls deformations of Green functors.*

It is the "homotopically meaningful" part that is most difficult here. The theory of derivations for Green functors are exactly the same as in the classical case. It is simplicial replacement which becomes more difficult. Here the failure of flatness for the symmetric powers of a general projective Mackey functor becomes most difficult: the simplicial replacement of a general commutative Green functor will not have an underlying projective simplicial Mackey functor. However, Goerss-Hopkins already provide an approach: by considering not commutative algebras but rather a simplicial replacement of an N_∞-operad which is level-wise discrete, we can avoid the failure of flatness.

The issue of equivariant grading arises here. Equivariant homotopy groups are naturally graded by the representation ring of G. Our chain complexes and the derived category of a Green functor should be similarly graded. We can easy consider the $RO(G)$-graded homology of a $\mathbb{Z}$-graded chain complex of Mackey functors: the V^{th} homology group of $C_\bullet$ is the group of chain homotopy classes of maps from the cellular chains on the V-sphere into $C_\bullet$. Connecting this back with the chains themselves and with the multiplicative structure is less obvious and will be essential when considering Green functors with norms.

Goal 14. *Work out the theory of $RO(G)$-graded chain complexes.*

The norm map is a map of multiplicative monoids of the same variance of the transfer. Algebraic geometry has already grappled with how to understand schemes together with a map from a multiplicative monoid: this is the notion of a "log-scheme". Copying this definition in <u>schemes</u> provides a way to describe the natural home in algebraic geometry for Tambara functors and other Green functors with norms.

Theorem 7. *There is a notion of a log-*<u>*scheme*</u> *giving a fully faithful embedding of Tambara functors into the category of log-*<u>*schemes*</u>*.*

In algebraic geometry, log structure most often describe singular or compactification points in moduli problems. This was why we used the log-étale topology to extend the derived structure sheaf of the moduli of elliptic curves. Here log structures should also describe those points in <u>Spec</u> where the localized Green functor fails to admit a Tambara structure because the norm maps "become singular". The other models for Spec of a Tambara functor also record just where quotients remain Tambara functors, but it is not clear how they compare.

Goal 15. *Determine the geometric content of the Tambara structure as a log-structure and compare this model to Nakaoka's.*

3.2 Landweber Exact Theories and $MU^{(G)}$

In the cases where we understand an equivariant rigidification of a naive G-spectrum produced by DAG, our understanding uses in an essential way the existence of a Real orientation, an equivariant map from $MU_{\mathbb{R}}$ to our spectrum. The theory of Real formal groups, explained quite cleanly by Hu-Kriz, shows that essentially everything we understand classically from work of Quillen works in the Real context [**?**]. Using the motivic Landweber exact functor theorem over $\mathbb{R}$ [**?**], we can deduce the Real form of the Landweber exact functor theorem. A first step in the progression from ordinary derived algebraic geometry to eDAG is to see that naïve Landweber exact theories lift.

Conjecture 16. *If R is a C_2-equivariant spectrum such that i_e^*R is Landweber exact and such that the C_2 action on the formal group of $i_e^*(R)$ is by the (-1)-series, then R is Real Landweber exact.*

Having a Real orientation bounds the size of the C_2-equivariant Hurewicz image to that of $MU_{\mathbb{R}}$, computed by Hu and already containing an impressive amount of the stable homotopy groups of spheres [**?**]. If, however, the spectrum R in the previous conjecture is the restriction to C_2 of a genuine commutative G-spectrum $\hat{R}$, then we can take the norm from C_2 to G to produce an $MU^{(G)}$-orientation of $\hat{R}$.

Goal 17. *Formulate and prove the G-equivariant version of the Landweber exact functor theorem for $MU^{(G)}$ and the appropriate equivariant notion of a regular sequence.*

All of this shows that $MU^{(G)}$ plays the role in eDAG that MU plays in DAG. Since computations with it are amenable to slice computations, we propose continued study of these spectra. In Hill-Hopkins-Ravenel, we determined a large family of differentials based only on elementary considerations of the geometric fixed points. This completely determined the orientability of sign representation spheres and their norms for $MU^{(G)}$-modules, and the differentials were sufficient to determine the required 256-fold periodicity by explicitly producing permanent cycles. However, even for C_4, we did not completely determine all differentials and extensions in the spectral sequence.

Hard Goal 18. *Determine the remaining differentials and the Mackey functor homotopy groups of $MU^{(G)}$.*

In particular, there is a natural subgoal of particular interest.

Goal 19. *Compute $\pi_{126}\Omega_{\mathbb{O}}^{hC_8}$ and see if it settles the remaining Kervaire case.*

The necessary computation may be closer in reach than initially thought. The 256-fold periodicity arises from the orientability of the representation $32\rho_8$:

$$\Sigma^{256} MU^{(G)} \xrightarrow{\simeq} \Sigma^{32\rho_8} MU^{(G)}.$$

If we knew instead that $16\rho_4$ was orientable (which requires only knowledge of the orientability of the unique 2 dimensional irreducible real representation of C_4), then the norm arguments in [**?**] would imply that $16\rho_8$ was orientable and that $\Omega_{\mathbb{O}}^{hC_8}$ was instead 128-periodic. This is an approachable C_4 computation.

I have several observations and conjectures about what happens in the slice filtration, and these make me optimistic that we can understand the slice spectral sequence for $MU^{(G)}$. The fact that this is a spectral sequence of Mackey functors imposes tremendous constraints. In particular, while there can be many extensions on the slice E_∞ term if we consider it as a spectral sequence of groups, there are often far fewer extensions when we consider this as a spectral sequence of Mackey functors. This was the key ingredient in the following theorem: the slice E_∞ term is easy to determine, and all of the relevant Ext groups actually vanish!

Theorem 8. *Multiplication by 4 annihilates the torsion in $\underline{\pi}_3 MU^{(G)}$ for all $n \geq 1$.*

In particular, though η and ν are both detected in the homotopy Mackey functors of $MU^{(G)}$, $4\nu = \eta^3$ is never so. Cursory computations near $\underline{\pi}_7$ suggest a similar story for σ. Assuming the Lubin-Tate spectra E_n can be made genuine equivariant commutative ring spectra, this shows that η^3 is almost always zero in the Hopkins-Miller higher real K-theories.

Conjecture 20. *For elements in the Hurewicz image, the order they have when they first appear in their expected (the Adams-Novikov) filtration is the largest order they ever achieve.*

Underlying Theorem 8 is an orientability condition: the proof includes a subresult that while no orientation class survives the slice spectral sequence, certain Euler classes times them can survive. The geometry of this kind of class is unclear.

Hard Goal 21. *Determine what kinds of bundles are oriented by $BP_G\langle n\rangle$.*

4 New Approaches in Topological Hochschild Homology

4.1 Topological Hochschild Homology and the norm

Angeltveit, Blumberg, Gerhardt, Lawson, Mandell, and I used the Hill-Hopkins-Ravenel norm technology to produce a new, conceptually simpler model for topological Hochschild homology [**?**]. The model arose from a simple observation: when viewed H-equivariantly, the standard cyclic bar model for $THH(R)$ is built out of $N_e^H(R)$. For commutative R, this simplifies further, and for expository reasons, we restrict to this case. This lead to a fairly straightforward definition for $THH(R)$ as an S^1-equivariant spectrum: the left adjoint to the forgetful functor from S^1-equivariant commutative rings to ordinary commutative rings. For sake of comparison, let $T(-)$ denote this left adjoint.

Theorem 9. *The S^1-equivariant commutative ring spectrum $T(R)$ is equivalent, relative to the family of finite subgroups, to* THH(R) *as a cyclotomic spectrum.*

This is a very weird functor in many ways. On free commutative ring spectra, this is straightforward: $T(\mathbb{P}(X)) \cong \mathbb{P}_{S^1}(S^1_+ \wedge X)$, where $\mathbb{P}_{S^1}$ is the free commutative ring spectrum functor. In particular, we see that for every ring spectrum R, the S^1-geometric fixed points of $T(R)$ is S^0.

Being a left adjoint, $T(R)$ has very nice formal properties. A useful application of this new model is a genuine S^1-equivariant model for THH of a Thom spectrum M_f generalizing the Blumberg-Cohen-Schlichtkrull non-equivariant model and generalizing the Madsen model for suspension spectra [**?**], [**?**].

Goal 22. *Describe a new model for* THH *of a Thom spectrum as a cyclotomic spectrum.*

Since left adjoints commute, we actually know exactly how to split up THH of a Thom spectrum equivariantly. The real heart of this goal is to understand the fixed point data giving us TR of Thom spectra. We have a filtration of an equivariant spectrum where the associated graded is the wedge over conjugacy classes of subgroups H of G of the homotopy orbits of the Weyl group of H acting on the H geometric fixed points:

$$Gr(X^G) = \bigvee_{[H]\subset G} EW_G(H)_+ \wedge_{W_G(H)} \Phi^H(X).$$

If X is a suspension spectrum, then this filtration splits: the Segal-tom Dieck splitting. For an equivariant Thom spectrum, we have no reason to believe that the filtration splits. However, the geometric fixed points of equivariant Thom spectra are again Thom spectra. The maps in the filtration are therefore interesting maps between Weyl homotopy orbits of Thom spectra. This is a fundamentally geometric computation which sheds light on foundational questions like the algebraic K-groups of MU.

4.2 Computations with equivariant THH and Thom Spectra

One of the most exciting aspects of the new equivariant model for THH is the relative version. Without any equivariance, it is easy to describe THH in the category of R-modules for a commutative ring spectrum R: form all of our smash products over R in the standard simplicial model for THH. Previous attempts to generalize this to the equivariant version of THH have broken down, because of the inability to understand what "equivariant commutative ring objects in R-modules" should be.

The norm functor describes exactly how to produce a relative version. If R is the commutative ring spectrum underlying a genuine S^1-equivariant commutative ring spectrum $\overline{R}$, then we have a natural map

$$T(R) \to \overline{R}$$

arising as the counit of the adjunction defining $T(-)$. By functoriality, if A is a commutative R-algebra, then $T(A)$ is a commutative $T(R)$-algebra. We then define

$$\mathrm{THH}^R(A) = T(A) \wedge_{T(R)} \overline{R}.$$

Since $T(-)$ takes values in cyclotomic spectra and since geometric fixed points commutes with smash products, we expect that the relative THH is cyclotomic. The problem is that we do not know that the geometric fixed points functor is homotopically meaningful on commutative ring spectra. This point will show up again below.

Conjecture 23. *If $\overline{R}$ is cyclotomic, then* $\mathrm{THH}^R(A)$ *is a cyclotomic spectrum.*

This construction depends in an essential way on $\overline{R}$, the equivariant refinement of R. Even for cases like $R = H\mathbb{Q}$, where the entire story is largely algebraic: the rationalized Burnside Mackey functor and the constant Mackey functor $\mathbb{Q}$ both refine $H\mathbb{Q}$.

If we are just given the commutative ring spectrum R, then there is a natural, functorial refinement $\overline{R}$ given by the push forward functor from trivial spectra to genuine G-spectra. The push forward functor i_* is the left-adjoint to the G-fixed points functor. Since commutative ring spectra have enough frees, and since we are looking at a left adjoint, it suffices to describe what i_*

does to a free commutative ring spectrum. Since left adjoints commute, i_* of a free commutative ring spectrum on X is the free *genuine* equivariant commutative ring spectrum on i_*X. In other words, we freely put in the norms. This is perfectly reflected in algebra: the free Tambara functor on a class in G/G is polynomial on norms (and some transfers on them). This lets us completely describe the underlying Tambara functor out of a commutative ring.

However, if R is not the zero sphere, then there is no reason to believe that this is a cyclotomic spectrum! In fact, it seems that it is *not* cyclotomic, satisfying instead a weaker condition that is nonetheless sufficient to define TC.

Conjecture 24. *If R is a commutative ring spectrum, then i_*R has the property that $\Phi^{C_n} i_*R$ is a retract of R as $S^1 \cong S^1/C_n$-equivariant commutative rings.*

Just as with the previous conjecture, this is true modulo a better understanding of the interaction of geometric fixed points with commutative rings. For motivation, if we consider the free E_∞ algebra on the suspension spectrum of X, then we get the suspension spectrum of the free algebra *in spaces* on X. Since geometric fixed points commutes with the suspension spectrum functor, we deduce that the geometric fixed points are the suspension spectrum on the C_n-fixed points of the free algebra on X. The tom Dieck splitting then provides the desired retraction.

5 Broader Impact

The broader impact of my project is via providing opportunities for mathematical discussion through major conference organization working with early-career mathematicians and underrepresented groups.

I was the lead organizer for the semester long program in algebraic topology at MSRI in Spring 2014, the first program in our field in over 25 years. This program had 65 algebraic topologists in residence at MSRI for various parts of the semester, and many more graduate students, post-docs, and researchers spent significant amounts of time there unofficially. Many more researchers came to the three workshops run in conjunction with the program.I was the lead organizer for one of these, the Introductory Workshop, a weeklong program designed to orient researchers, especially younger researchers and people in other fields, by having experts give broadly focused, expository talks highlighting their area of expertise. Over 240 mathematicians registered for this workshop, making it one of the largest held at MSRI.

Prior to MSRI, I coorganized "Equivariant homotopy and algebraic K-theory" at Banff in 2012 and the "Virginia Conference on Algebraic Topology" in 2012. I intend to continue organizing conferences during the period covered by this grant. With Blumberg, Gerhardt, and Lawson, I have applied for a Banff workshop in equivariant derived algebraic geometry. With Blumberg, Gerhardt, and Ormsby, I am applying for an AIM workshop in equivariant derived algebraic geometry.

As part of the broader MSRI program, I coorganized two summer school in algebraic topology for graduate students, both of which focused on attracting a diverse collection of graduate students from a wide range of institutions. MSRI asked us to reprise it in a joint program with CIMAT in Mexico in 2014. Both summer schools focused on 5 lecture series by prominent mathematicians and on approximately 20 talks given by participants on foundational papers in the field. We also had 2 "job skills" panel discussions, allowing students the opportunity to ask questions on everything from applying for jobs (both in the US and abroad) to finding problems and funding.

In the past two years, I have given short courses at 4 programs targeting graduate students. In 2013 at the Vietnam Institute for Advanced Study in Mathematics, I gave a two week master class on the Kervaire Invariant One problem. In 2013, in Copenhagen, I gave a week long master class on equivariant homotopy. At the MSRI summer schools in algebraic topology in 2013 and 2014, I gave a series of four lectures on equivariant homotopy. I have compiled extensive notes and problems for these summer schools, and I plan to turn them into a book on equivariant homotopy theory stressing computational aspects of the field.

I also currently have three graduate students: John Berman, Peter Bonventre, and Scott Slinker. I recently had two students graduate: Kristen Mazur and Carolyn Yarnall. They both currently hold three year postdocs at liberal arts colleges.

6 Results from Prior NSF Support

Intellectual Merit

From June 2009 through the end of May 2012, I was supported by NSF grant DMS-0906285: "Computations in Classical Chromatic Homotopy Theory, Algebraic K-Theory, and Motivic Homotopy" for $100,886. The first three Kervaire papers below were also supported by a post-doc from 2009-2010, sponsored by the Chas-Hopkins-Stoltz-Sullivan-Teichner FRG, NSF grant DMS-0757293: "FRG: Collaborative Research: How the Algebraic Topology of Closed Manifold Relates to Strings and 2D Quantum Field Theory ". I received $60,000 and health insurance.

During this grant, [?], [?], [?], [?], and [?] were all written and published. The survey article about the Kervaire proof included several independent arguments for key parts of the solution, providing additional intuition and other approaches. The slice primer included a new approach to slice computations and several conjectures. The Ext computation introduced my "ρ-Bockstein spectral sequence", the primary tool for the motivic Adams spectral sequence over $\mathbb{R}$.

From May 2012 to May 2015, I have been supported by NSF grant DMS - 1207774: "Computations in Equivariant Homotopy and Algebraic K-Theory" for $293,000. All of the work which has been produced during this time has been explained in the projects described above. The papers [?], [?], and [?] have appeared, and the Talbot proceedings volume I edited [?] has been accepted. The article [?] was submitted to Advanced in Mathematics and [?] was submitted to Inventiones. The papers [?] and [?] have been completely written and should be submitted prior to review. Finally, the preprints [?] and [?] are on the arXiv.

Finally, [?] is submitted to Annals. It has been completely rewritten twice: once during DMS-0906285 and then again during DMS-1207774.

While supported by these grants, I gave a total of 29 talks domestically, including as an invited speaker at an AMS sectional meeting. I also gave 41 lectures at international conferences and summer schools, including most excitingly as an invited speaker at the ICM.

Broader Impact

The broader impact of the prior work was largely described in more detail above. While supported by these grants, I organized four international conferences, two summer schools targeting graduate students, and the semester long program at MSRI in algebraic topology. I gave short courses at five summer schools. I also graduated two students and became the advisor of three more.